A GENTLE INTRODUCTION TO CALCULUS

Please don't panic

Questions

I'm quite open to questions during the presentation. But…

- We've got a *lot* to get through, and a limited time to do so in, so if things even *start* to get out of hand, I may have to be firm and cut off the discussion.
- There are over 50 slides, which allows me at most 1-2 minutes per slide, even without questions. Time will be tight.
- Again, for time constraint reasons, I'll accept only questions that boil down to "I don't understand", "could you go over that again" and so on. If any other types of questions are asked, I'll have to be curt and say that we have to defer the question.
- When the presentation is over, if we have any time left, then the field will be open to any questions.

What is Calculus?

The term Calculus comes from the Latin for "pebble" and is, of course, related to the word "calculate".

There are several main types of calculus, each with sub-genres. For this introductory presentation, we'll talk exclusively about Differential Calculus, which finds the slope (*) of a curve.

Note: I won't emphasize the following, but you might notice from the graphs that follow that the slope also indicates the rate of change of a function(*) at a given point. The larger the slope, the faster the function value will be changing at that point.

(*) We'll define the terms *slope* and *function* shortly.

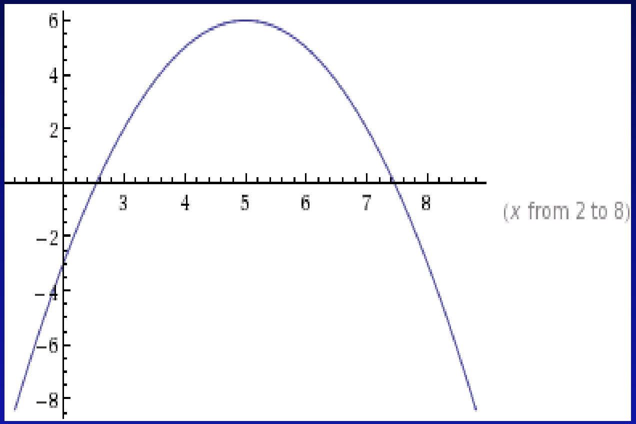
Where We're Going

Nature seems to have the property that it tries to either maximize or minimize some property.

For example, Fermat's Principle states that the path taken between two points by a ray of light is the path that takes the least time.

If we graph an equation for a physical process, we'll often see that it climbs to its maximum value, then decreases. Or descends to its minimum value then increases.

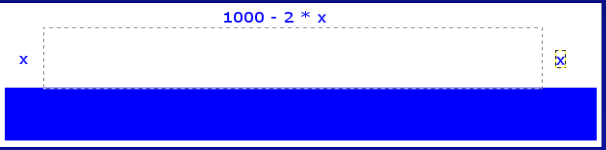
Our goal with calculus is to find the value that maximizes (or minimizes) its value.



Our Two Sample Problems

We have limited time for this lecture, so we're going to solve only two problems.

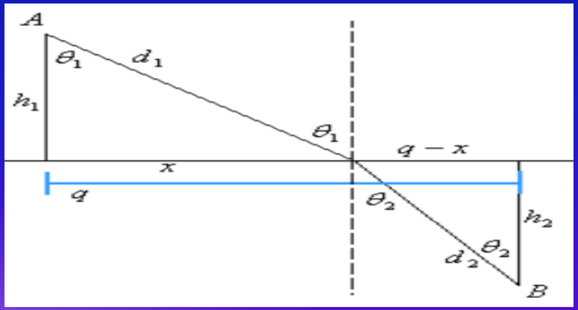
The first is a simple problem. You have a certain amount of fencing and you want to build a rectangular pen of maximum area against the side of a river. That fourth side doesn't need fencing. What should the dimensions of the pen be?



The second is much more interesting.

Imagine you're a lifeguard on a beach and you hear a cry for help from someone in the water. You can run faster on land than you can swim in water. What can you tell me about the path you should take to get to the distressed swimmer as quickly as possible?

But I should warn you. I've also strewn a few homework problems throughout the text!



What You Need To Know

Simple high-school algebra

Simple graphing

How to calculate the slope of a line

And that's all!

Our Approach

- We'll take a simple curve (a parabola) and try to find the slope of the curve at a particular point.
- We'll get closer and closer to our result by a series of closer and closer approximations.
- Intuitively this will give us our answer.
- But then we'll generalize. Instead of finding the slope of a *parabola* at a *specific* point, we'll develop a formula that will let us find the slope of *any* curve at *any* point.
- Once we have that, we'll apply our formula to a simple example.
- Then we'll tackle a bigger problem, which will lead to a surprising result!

- But first we'll take a refresher course in some of the simple math we'll need along the way.

Basic Concepts: Multiplying Expressions

Later on we'll want to multiply the expression (x + h) by itself.

(x + h) * (x + h)

Note: To avoid confusion with a variable name of "x", and the multiplication symbol (also "x"), I've used "*" to indicate multiplication.

In high school you may have been taught the FOIL (First, Outside, Inside, Last) rule.

First:	x * x	(**x** + h) * (**x** + h)
Outside:	x * h	(**x** + h) * (x + **h**)
Inside:	h * x	(x + **h**) * (**x** + h)
Last:	h * h	(x + **h**) * (x + **h**)

Add them together and you get

x*x + x*h + h*x + h*h

= $x^2 + 2xh + h^2$

Let's Try It

We claim that $(x + h) * (x + h) = x^2 + 2xh + h^2$

Well, consider 7 * 7 = 49.

But we can also write this as (3 + 4) * (3 + 4), where x = 3 and h = 4.

So	$x^2 = 3 * 3$	=	9
	$2xh = 2 * 3 * 4$	=	24
	$h^2 = 4 * 4$	=	16

			49

Amazing. It actually works!

Basic Concepts: The Slope of a Line

Imagine two points on a line, say (3, 5) and (10, 26).

How fast is that line increasing?

The rise: difference in the y-coordinates (26 – 5) = 21.

The run: difference in the x-coordinates (10 – 3) = 7.

The slope is defined as “the rise over the run”.

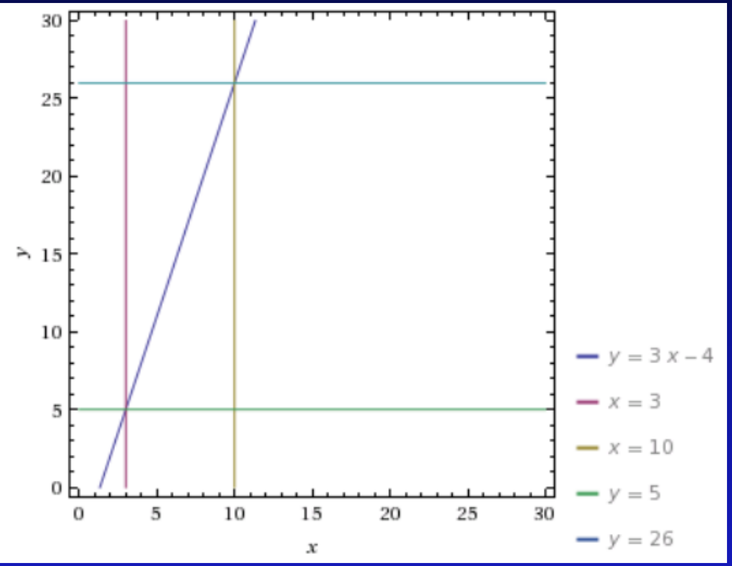
Thus the slope is 21 / 7 = 3.

In general, if the points are (x_1, y_1) and (x_2, y_2)...

rise = $y_2 - y_1$

run = $x_2 - x_1$

slope = $(y_2 - y_1) / (x_2 - x_1)$



Basic Concepts: Functions

A function is a rule about how to calculate something.

It has a name (e.g. “f”, but could be, say, “sin” for the trig function sine).

Sometimes we use a symbol for it (e.g. √), but here we’ll always use names.

It has one or more parameters (values).

We write it as, say, “f(x)” (“eff-of-ecks”) or “g(x, y)” (“gee-of-ecks-comma-wye”)

We must define the rule. For example, a function named “f” that takes a single parameter and calculates its square would be “f(x) = x * x”.

Examples for f(x) = x * x:

$f(3) = 9$

$f(4) = 16$

$f(1.5) = 2.25$

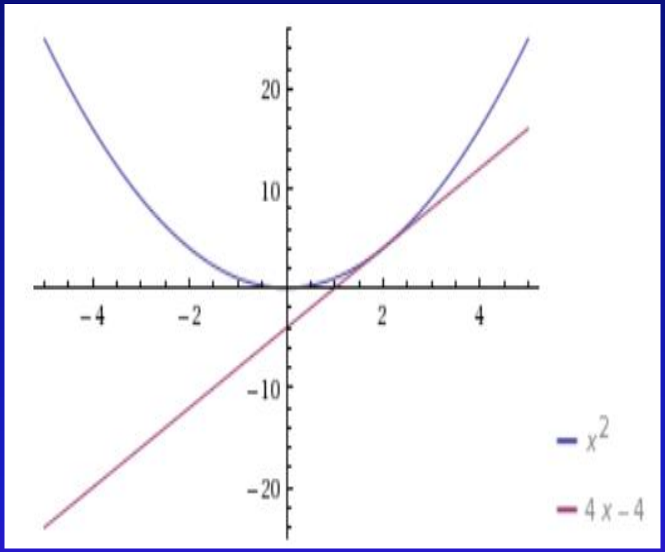
$f(x + h) = x^2 + 2xh + h^2$

Going Off On A Tangent

A tangent is a straight line that just "kisses" the graph of a function at a point.

For example, here's a tangent to the parabola $y = x^2$ at the point (2, 4).

By definition, the slope of a curve at a point is the slope of the tangent at that point.



Note: the scale on the two axes is different. We'll do this in most of our diagrams to help focus on some aspects of the graph, at the admitted cost of some distortion. So while the tangent above looks like it has a slope of about 1, it actually has a slope of 4.

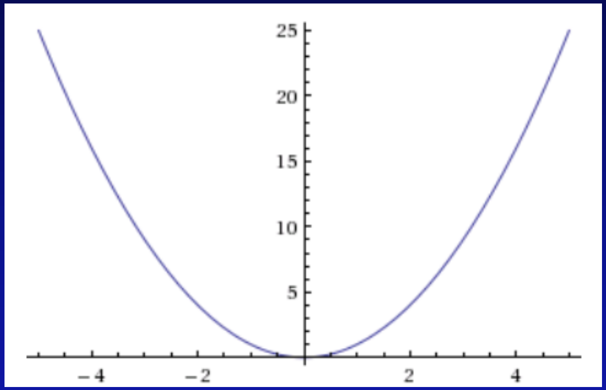
Why Do We Care About Tangents?

Functions normally have one or more extreme points, either maximums or minimums.

Nature seems to favor either maximizing or minimizing various quantities (e.g. energy, time, etc).

At an extremum, the slope of the tangent is zero!

So if we can find where on the graph the slope of the tangent is zero, we can probably find some interesting physics.



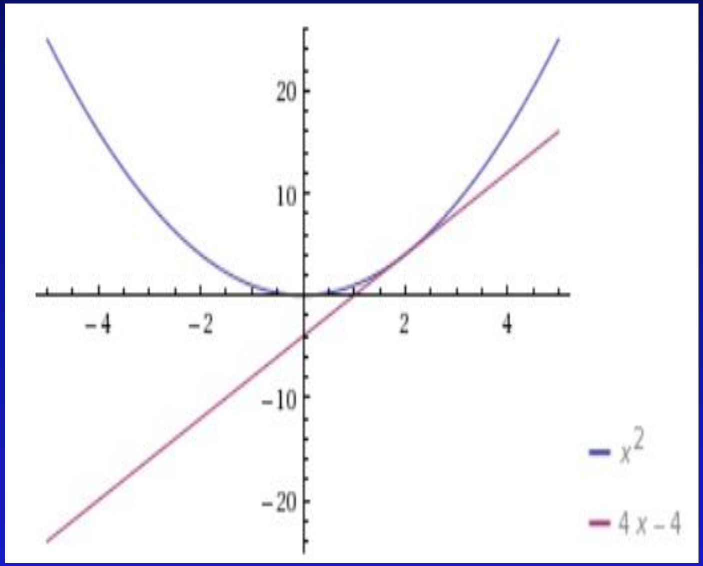
Differential Calculus

The main job of differential calculus is, given a function, find the slope of the tangent at a general point on the curve.

For our parabola example, the slope of the tangent at the point (2, 4) is 4.

As we'll see, the slope at (3, 9) is 6.

And as we can see intuitively, the slope is zero at (0, 0). But can we *prove* this?



Why is y = f(x)?

Before we go any further, let's be clear about something.

Sometimes I'll write a point as (x, y).

Other times I'll write (x, f(x)).

Which is correct? And the answer is that they're the same.

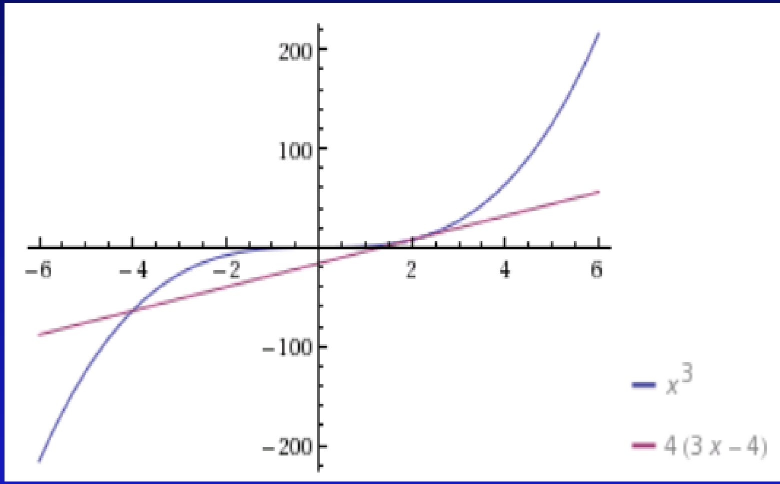
When we graph something, we normally pick an x value, then calculate the corresponding y value.

But how do we do that? Answer: We're drawing the graph of some particular function that for each value of x there's a corresponding y value calculated as y = f(x).

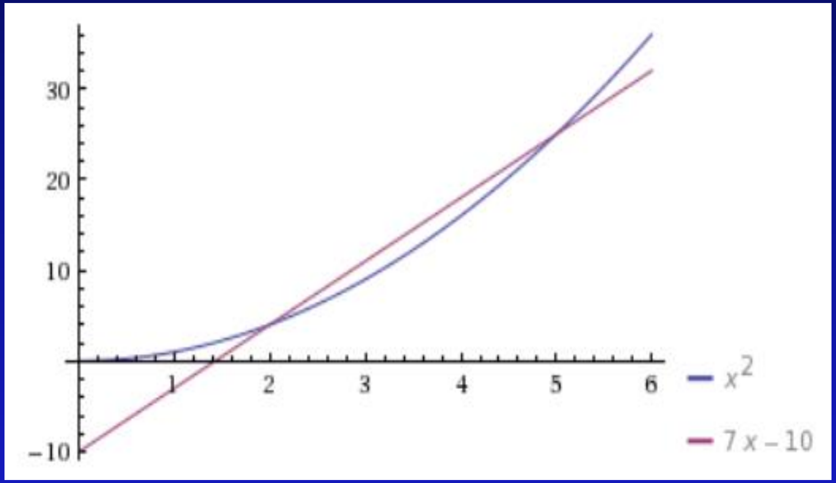
So in general, we can (and will) use *y* and *f(x)* interchangeably.

Secant Ye Shall Find

A tangent intersects a graph at a single point (and maybe others).



Whereas a secant intersects a graph in two places (and maybe more).



Note that this is the graph of $y = x^3$ and the scale is very much distorted. Otherwise we wouldn't be able to see where the tangent intersected the curve to the left.

A Pivotal Decision

In what follows, we'll develop a way to find the slope of the tangent (the derivative) to the function $f(x) = x * x$ at the point (2, 4).

That will be our pivot point. We'll see exactly what this means in the next slide.

Secants Lead to Tangents

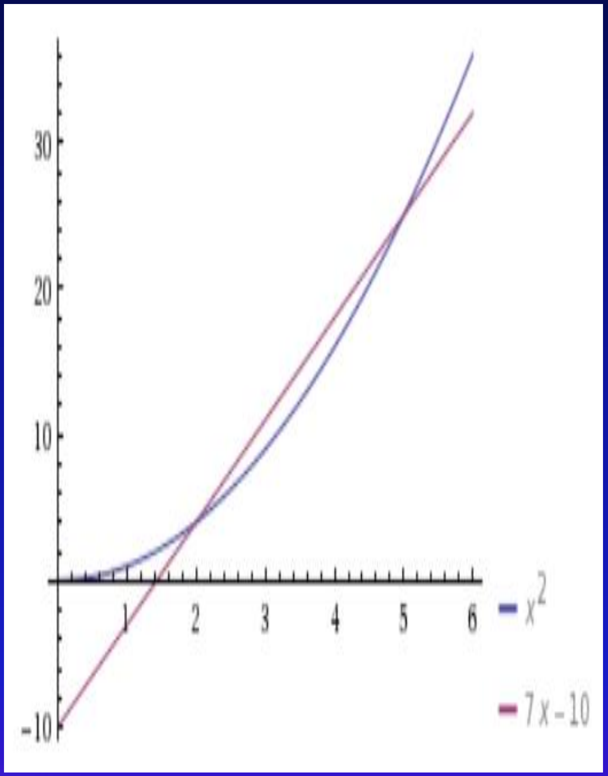
Suppose we want to calculate the slope of the tangent to function f(x) at a given point. This will be our pivot point.

Draw a secant through the pivot point to a nearby secondary point on the graph. Calculate its slope. This will be a first approximation to the slope of the tangent.

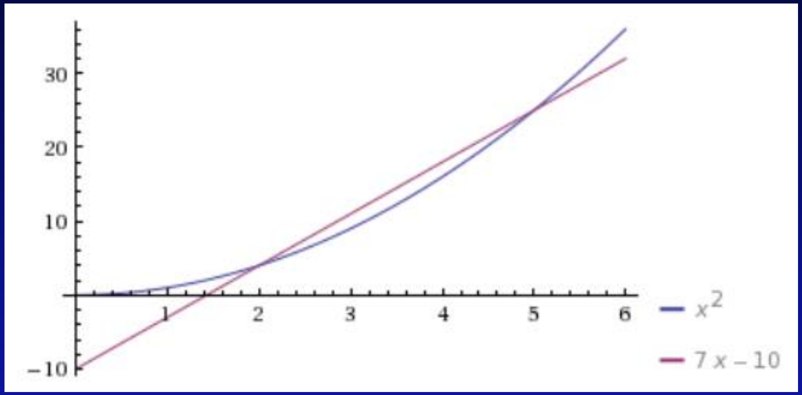
Keeping the same pivot point, move the secondary point nearer to the pivot point to create a new secant. Calculate its slope. This will be a better approximation.

As you get closer and closer to the pivot point, the slope of the secant is a better and better approximation to the slope of the tangent.

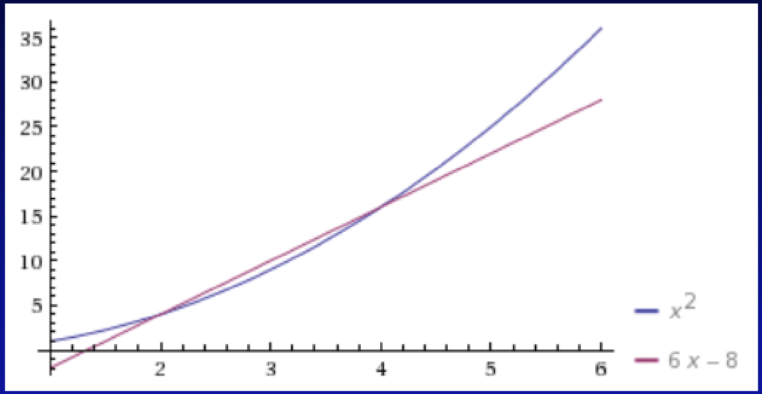
By the time you reach the pivot point, the secant will have become the tangent and you've got your answer.



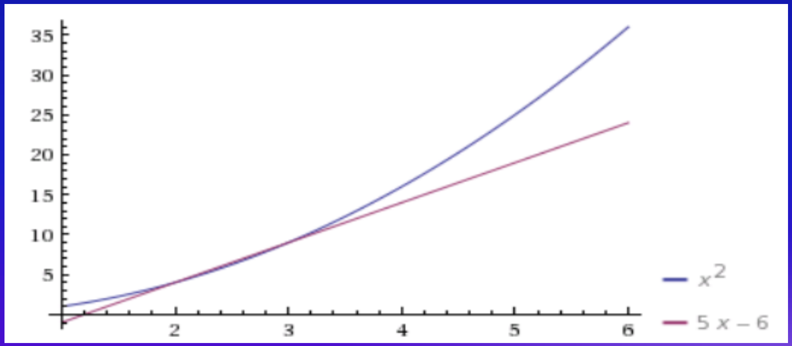
So Let's Watch This In Action



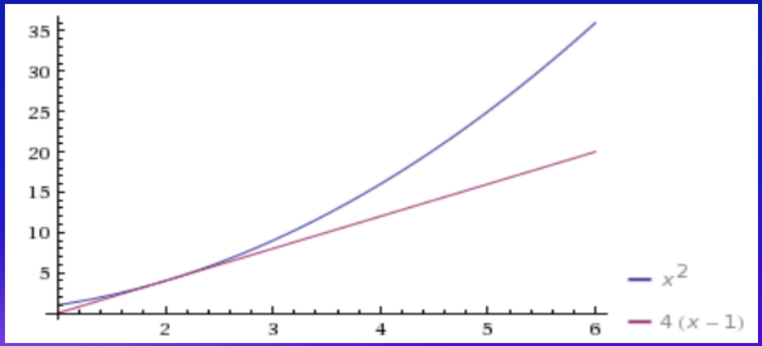
Secant to (2+3, 25)
Slope = (25 – 4) / (5 – 2) = 7



Secant to (2+2, 16)
Slope = (16 – 4) / (4 – 2) = 6



Secant to (2+1, 9)
Slope = (9 – 4) / (3 – 2) = 5



Tangent! Slope = 4 (intuitively)

Did You Notice A Pattern?

From the previous slide, let's look at the slope of the secant based upon our offset.

x_1	$2x_1$	h	$2x_1 + h$	Slope
2	4	3	7	7
2	4	2	6	6
2	4	1	5	5

I see a pattern there. The slope of the secant is $2x_1 + h$.

Can we prove this?

And as h becomes smaller and smaller, the slope of our tangent gets closer to $2x_1$ (which in this case gives 4).

And maybe the slope of the tangent at *any* point x on a parabola becomes closer and closer to 2x.

But can we prove this???

OK, Fasten Your Seat Belts

The next couple of slides is where we actually do a bit of elementary algebra.

And again, please don't panic!

Doing it in General

We want to calculate the slope of the secant from the point (x, f(x)) to a nearby point for the function f(x) = x * x.

In our previous example, this was at the point (2, f(2)) = (2, 4).

So the starting (pivot) point for our secant will, in general, be

(x, f(x))

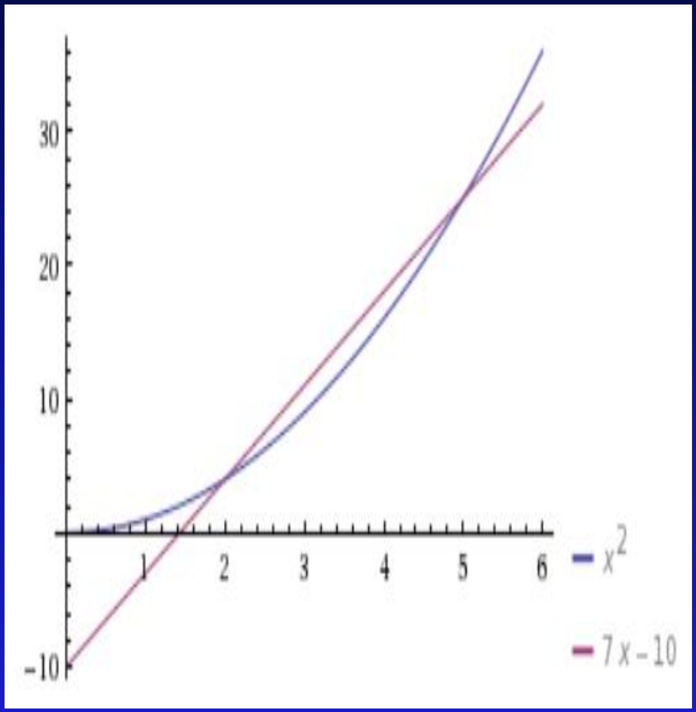
Now move to a new x coordinate at some distance to the right (or left). We'll call this distance *h*. So our second point will be

(x + h, f(x + h)).

So with x = 2 and (say) h = 3, our second point will be at

(2 + 3, f(2 + 3)) = (5, f(5)) = (5, 25)

If we started at the point (10, 100), our second point would be (10 + h, f(10 + h)).



Slope of the Secant, in General

So we've got two points and they define a straight line. What's its slope?

From the previous slide, two points on a secant are

(x, f(x))   and   (x + h, f(x + h))

So the rise is $y_2 - y_1$ = f(x + h) – f(x)

The run is $x_2 - x_1$ = x + h – x = h

So the slope of the secant is the rise over the run

$$\frac{f(x+h)-f(x)}{h}$$

And that's the general formula for the slope of a secant for any curve f(x).

Slope of the Secant for Our Curve

Let's plug some details into $\frac{f(x+h)-f(x)}{h}$ for $f(x) = x^2$

$f(x+h) = (x+h) * (x+h) = x^2 + 2xh + h^2$

$f(x+h) - f(x) = 2xh + h^2$ The rise

h The run

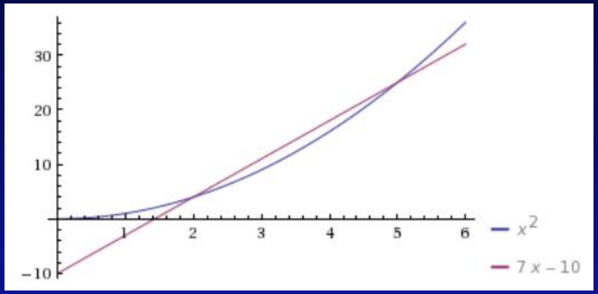
Thus $\frac{f(x+h)-f(x)}{h}$ = $\frac{2xh+h^2}{h} = 2x + h$

Important note! We've just divided by h. Hold that thought. We'll need it shortly.

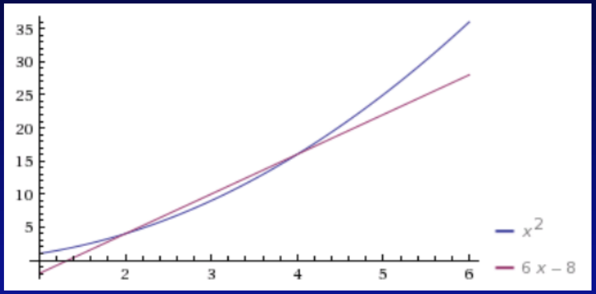
So the slope of the secant on our parabola from the point (x, f(x)) to the point *h* units over is

$$2x + h$$

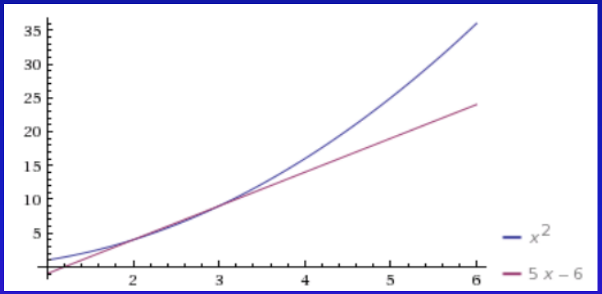
Looking Back a Few Slides



Secant to (2+3, 25)
Slope = (25 – 4) / (5 – 2) = 7



Secant to (2+2, 16)
Slope = (26 – 4) / (4 – 2) = 6



Secant to (2+1, 9)
Slope = (9 – 4) / (2 – 1) = 5

So we see that our formula

Slope of secant = 2x + h

works!

For example, if h = 3, the slope = 2 * 2 + 3 = 7

Finding Tangents is Secant Nature

The slope of the secant for $f(x) = x^2$ between the points $(x, f(x))$ and $(x + h, f(x + h))$ is $2x + h$.

As we get closer to our original coordinate x, h becomes smaller and smaller.

In the limit, h approaches 0 and we're left with just 2x.

For the function $f(x) = x^2$, the slope of the tangent at any point x is simply $2 * x$

We say "The derivative of x^2 is 2x"

And the process of finding a derivative is called differentiation

Important Note

We had the formula for the slope of the secant as 2x + h.

Why didn't we just set h = 0 and get 2x?

Because when we derived the formula, at one step we divided by h. And if we decide later that h was equal to zero, then that invalidates our whole argument.

You ***can't*** divide by zero!

To get around this problem, mathematicians developed the concept of a *limit*.

We don't have time to go into the slightly subtle details of exactly what this is. Those interested can find various articles on the web. See the References section at the end of this presentation for places to start. But intuitively, it means that as h gets closer and closer to zero, our calculation (2x + h) gets arbitrarily close to 2x, and that's going to be our answer.

So in the rest of this presentation, I'm just going to pretend that h is zero. You and I both know that this is wrong, but it's close enough for now.

Notation

We often write $f'(x)$ ("eff prime of ecks") for the derivative. This is the notation used by Sir Isaac Newton.

So $f'(x^2) = 2\mathrm{x}$.

Gottfried Leibniz (who independently invented calculus) came up with the (slightly strange) notation

If y = f(x), then he wrote $\frac{dy}{dx}$ for the derivative.

But beware. Although this looks like a fraction, it isn't.

To avoid confusion, we'll use the $f'(x)$ notation.

Other Derivative Formulas

We won't prove these, but mathematicians have developed the formulas for the derivatives of many common (and many uncommon) functions.

Make note of the formulas for the derivatives of

c * f(x) (where c is a constant)

f(x) ± g(x)

$\sqrt{x^2 + a^2}$ (*a* constant)

$\sqrt{a^2 + (q - x)^2}$ (*a*, *q* constant)

We'll use these later.

Note: Rules 1-5 are all special cases of Rule 6. And rule 6 is a special case of rule 12.

Homework – Derive the formula for the derivative of $f(x) = x^3$

Rule #	Function Rule (y = …)	Derivative
1	c (any constant e.g. 6.8)	0
2	x	1
3	x^2	2 x
4	x^3	$3 x^2$
5	x^n	$n x^{n-1}$
6	cx^n	$c * n x^{n-1}$ (e.g. $7.3 * n x^{n-1}$)
7	sin(x)	cos(x)
8	cos(x)	-sin(x)
9	f(x) ± g(x)	f'(x) ± g'(x)
10	f(x) * g(x)	f'(x)g(x) + f(x)g'(x)
11	f(x)/g(x)	$(f'(x)g(x) - f(x)g'(x)) / g^2(x)$
12	$c * f(x)$	$c * f'(x)$
13	$\sqrt{x^2 + c^2}$	$\frac{x}{\sqrt{x^2 + c^2}}$
14	$\sqrt{c^2 + (q - x)^2}$	$\frac{-(q - x)}{\sqrt{c^2 + (q - x)^2}}$

Let's Work Out an Example

Suppose we have a function whose formula is

$1000 * x - 2 * x^2$

This is of the form f(x) – g(x), where

$f(x) = 1000 * x$

$g(x) = 2 * x^2$

So the derivative of this is

$f'(x) - g'(x)$ *(rule 9)*

$f'(x) = 1000$ *(rule 6 / 12)*

$g'(x) = 2 * 2x = 4x$ *(rule 6 / 12)*

So the derivative of our full expression is

$1000 - 4x$

The slope of the tangent to the curve

$f(x) = 1000 * x - 2 * x^2$

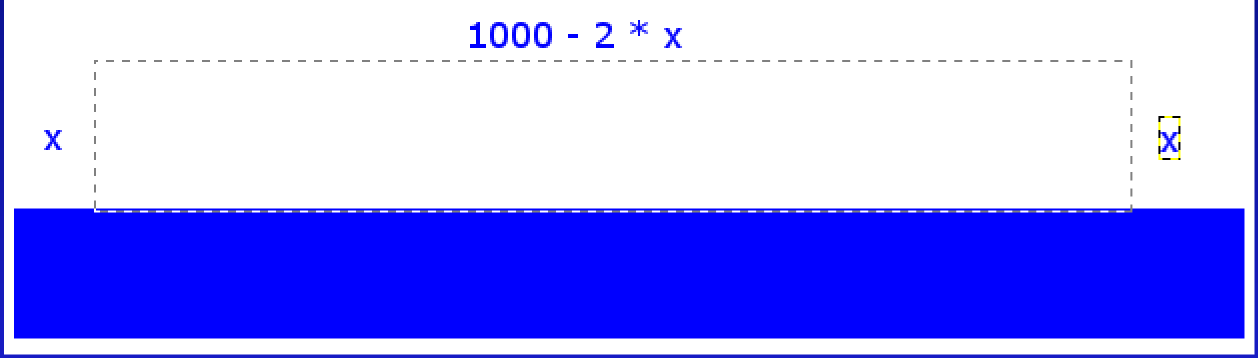
at point x is $1000 - 4x$

Rule #	Function Rule (y = ...)	Derivative
1	c (any constant e.g. 6.8)	0
2	x	1
3	x^2	2 x
4	x^3	$3 x^2$
5	x^n	$n x^{n-1}$
6	cx^n	$c * n x^{n-1}$ (e.g. $7.3 * n x^{n-1}$)
7	sin(x)	cos(x)
8	cos(x)	-sin(x)
9	f(x) ± g(x)	f'(x) ± g'(x)
10	f(x) * g(x)	f'(x)g(x) + f(x)g'(x)
11	f(x)/g(x)	$(f'(x)g(x) - f(x)g'(x)) / g^2(x)$
12	$c * f(x)$	$c * f'(x)$
13	$\sqrt{x^2 + c^2}$	$\frac{x}{\sqrt{x^2 + c^2}}$
14	$\sqrt{c^2 + (q - x)^2}$	$\frac{-(q - x)}{\sqrt{c^2 + (q - x)^2}}$

OK, a Simple Example

Imagine you own a plot of land next to a river. You have 1000 meters of fencing and you want to create a rectangular 3-sided pen to keep out animals. The river will be the 4th side of the pen and does not need fencing.

What (rectangular) dimensions should the pen be to maximize its area?



Let's have some guesses here before going on to the next slide…

Fencing

Since the pen is rectangular, let the two sides perpendicular to the river be x. This will be the height of our rectangular pen.

So the third side will have length 1000 – 2 * x.

Thus the area will be

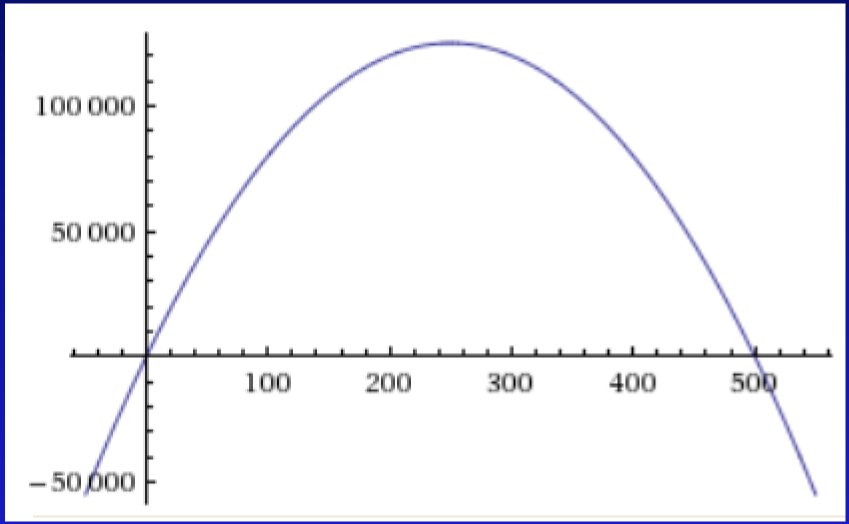
$$f(x) = x * (1000 - 2 * x)$$

$$f(x) = 1000 * x - 2 * x^2$$

This is a parabola.

When we plot this, we can see intuitively that the maximum area is somewhat more than 100,000 square meters, for an x value of around 250 meters.

But note that at the apex, the slope of the tangent is zero. Aha!



Piece of Cake!

So if

$f(x) = 1000 * x – 2 * x^2$ (The area of our pen)

Then the derivative (slope of the curve at point x) is

$f'(x) = 1000 – 2 * 2 x = 1000 – 4x$

To find the maximum, see where the derivative (the slope of the tangent) becomes 0.

If $1000 – 4x = 0$, then

$1000 = 4x$

$250 = x$ (the perpendicular length)

And the third side would be 1000 – 250 * 2 = 500 meters.

Thus we should make a pen of dimensions 250 * 500 for a maximum area of 125,000 square meters.

Un morceau de gateau!

A Slightly Closer Look

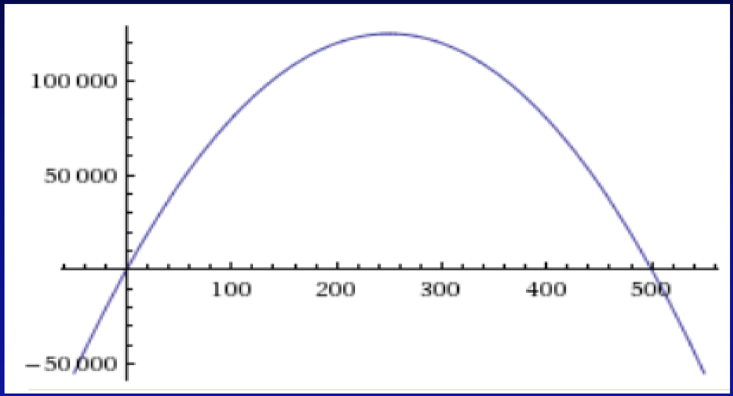
So we know that the slope (derivative) is 1000 – 4x. So let's take a look at some values for our original function, $1000x - 2x^2$

In the table to the lower right, we see that going from 0 towards 250, the slope is positive, meaning the original function is increasing over that range, but slowing down. (As we mentioned early on, the slope shows the *rate of change* of the function.)

But past x = 250, the slope is negative, showing that the function (the area) is declining. And ever faster.

Which perfectly matches our visual concept of the graph.

Phew! It works! :)



x (height)	Slope	Area
0	1000	0
150	400	105,000
200	200	120,000
250	0	125,000
350	-400	105,000
400	-600	8,000
500	-1000	0

Review

Let's step back and look at what we've done.

- We've approximated a tangent by a secant.
- We've developed a surprisingly simple formula for the slope any secant.
- We've tweaked the secant (by lowering 'h') to give better and better estimates to the tangent.
- In the limit ($h \to 0$) this gives us the slope of the curve at any point.
- Interesting physics (a maximum or minimum) can often be found when the slope is 0.
- So we take our formula for the slope (the derivative of our given function), set it equal to 0, and solve for x.
- And that's our maximum (or minimum)!

Homework – Is It Consistent?

To give you exercise in setting up and solving these problems, consider the following…

Our approach to the pen was to assume the height of the rectangle was x, which led to a width of 1000 – 2x.

Redo this analysis assuming the top of the rectangle is of width *x*. What are the lengths of the other sides, what's the new formula for the area, and do we come to the same conclusion? (We'd better!)

Homework – Optimum Shape

Assume we must create a rectangular fenced-in area with L units of fencing, and have it enclose as large an area as possible.

What must its dimensions be?

Thus what is the maximum rectangular area you can enclose with L units of fencing?

Just to be perfectly clear, non-rectangular shapes such as circles aren’t admissible shapes for this problem. Also, the fencing must cover all 4 sides (no river in this case).

Let's Take a Break

See you back here in 5 minutes. OK?

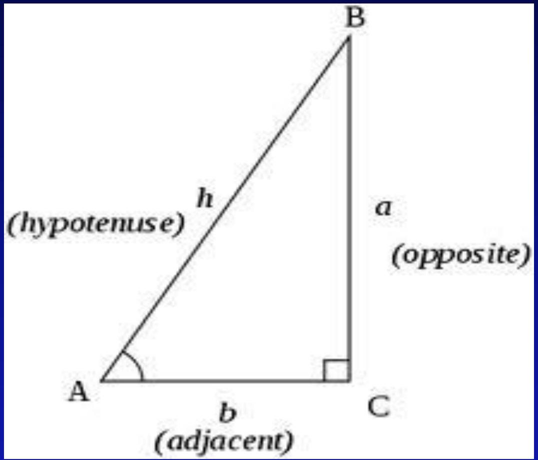
Two More Things to Know
Pythagorean Theorem

We all know this one, right?

In a right-angled triangle, *the square on the hypotenuse is equal to the sum of the squares on the other two sides.*

$$h^2 = a^2 + b^2$$

or $$h = \sqrt{a^2 + b^2}$$



Please note: This *h* has nothing to do with the *h* we've been using up to this point. Even with Greek and other alphabets, they have so many concepts that at times they've used the same letter for totally different ideas. So please don't be confused.

Two More Things to Know
Just a Hint of Trigonometry

By definition, in a right-angled triangle, the sine of angle A is defined as the ratio of the length of the side opposite the angle divided by the length of the hypotenuse.

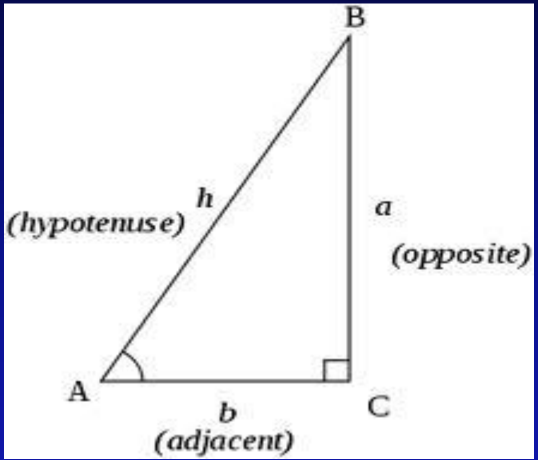
$$\sin A = a / h$$

Similarly, $\sin B = b / h$.

Every angle has a corresponding numeric value called the sine. Consult a scientific calculator app on your phone.

Note: It doesn't matter what size the right triangle is. For a given angle, the ratio of the two sides is always the same.

There are names for the other ratios (e.g. b / a, etc), but we won't need these.



Angle (θ)	sin θ
0º	0
30º	0.5
45º	$\frac{1}{\sqrt{2}} = 0.707\ldots$
60º	$\frac{\sqrt{3}}{2} = 0.866\ldots$
90º	1

OK, Now Onto Something Challenging

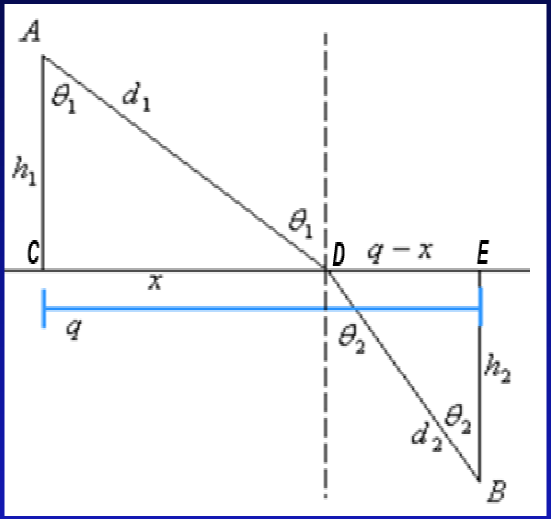
Suppose you're a lifeguard on a beach, sitting in your high chair 50 meters from the water (We'll call this point A). Suddenly you hear someone in the water calling for help. They're (say) 300 meters to the east (distance 'q') and 30 meters from shore. This is point B. You need to get to the person as quickly as possible.

Suppose you can run 5 meters/second on land and swim 1 meter/sec. What can you tell about the path the lifeguard must travel?

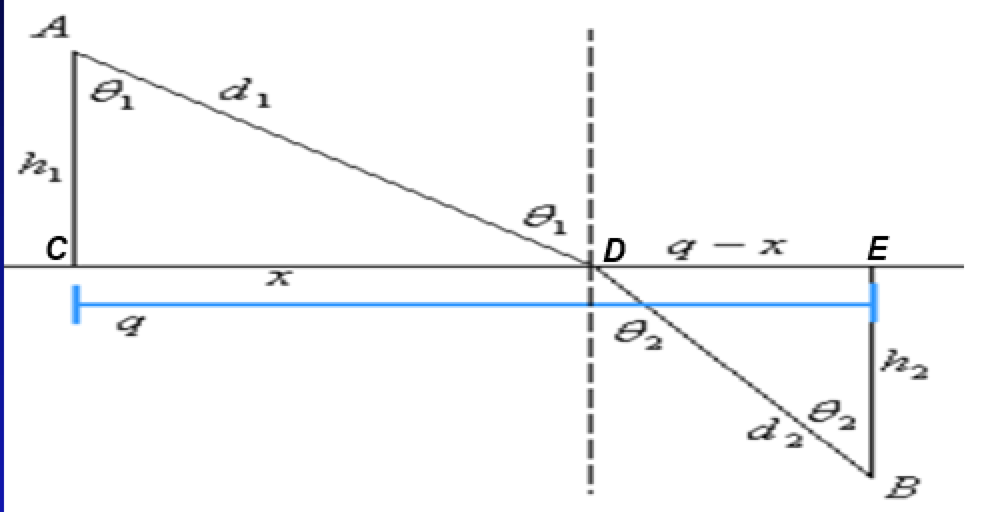
We'll need the following later: he's going to leave his station at some angle. We'll call this θ_1. (The Greek letter *theta*, and pronounced as either "thay-ta-sub-one", or perhaps just "thay-ta-one".)

In general he may change direction when in the water. We'll call that angle θ_2.

Note: There are two θ_1's in the diagram. By elementary geometry, they're equal. So are the two θ_2's.



Which Way Did He Go, George?



Does he

a) Run the 50 meters directly to the water, then start swimming?

b) Run diagonally to the shore directly in front of the swimmer and start swimming?

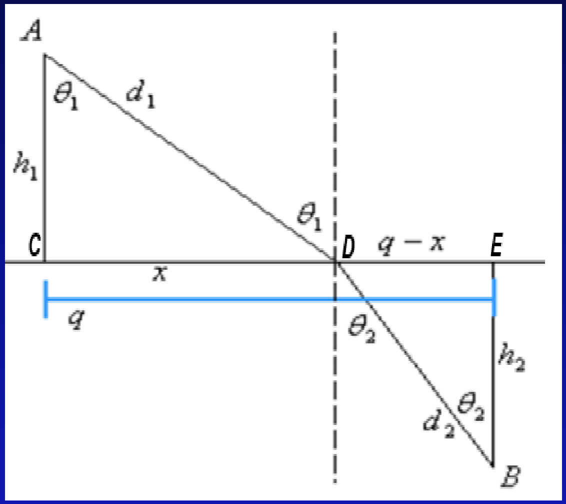
c) Run diagonally straight for the swimmer and enter the water?

d) Something else?

Roadmap

How are we going to approach solving this? Actually it's pretty straightforward.

- The total distance taken will be from A to D, then from D to B. We'll use the Pythagorean Theorem to get the length of each of the two segments.
- The time taken will be AD divided by his running speed on land, added to DB divided by his swimming speed.
- We'll differentiate that, set it to zero, then solve.
- Easy peasy!



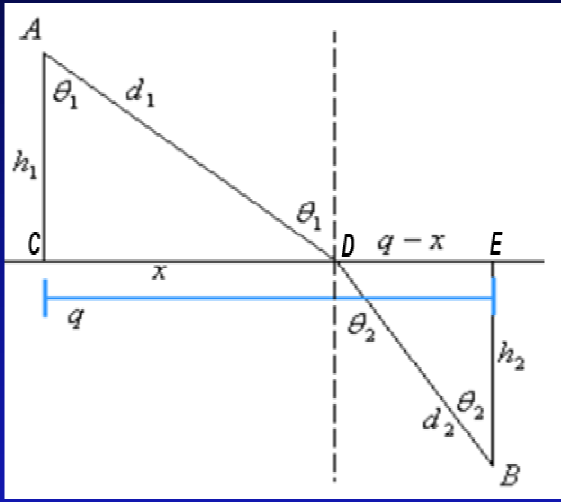
Setting Things Up - 1

- The lifeguard's path will be from A to D (on land), then from D to B (in the water).
- We'll call his speed on land S_{land}, and in the water, S_{water}. Note that both these (for this problem) are *constants*. We'll need to remember this when we differentiate.
- He'll enter the water at point D, some distance *x* east of him. We know the total horizontal distance to B (in our example, 300 meters) that we'll call *q*, so the distance from D to E is *q* – *x*.
- AD is the hypotenuse of right triangle ACD, so its length is $d_1 = \sqrt{CD^2 + AC^2} = \sqrt{x^2 + AC^2}$. Again, for this problem, AC (and thus AC^2) is a constant.
- And thus the time he takes to go from A to D is

$$t_1 = \frac{\sqrt{x^2 + AC^2}}{S_{land}}$$

- Similarly, the time he spends in the water is

$$t_2 = \frac{\sqrt{EB^2 + (q - x)^2}}{S_{water}}$$

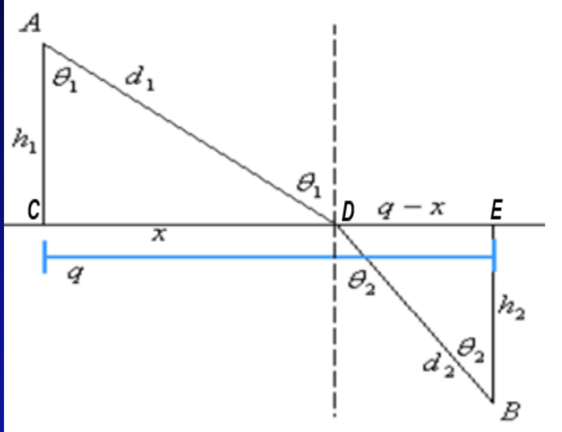


Setting Things Up - 2

- So the total time it takes to get to the distressed swimmer is

$$\frac{\sqrt{x^2 + AC^2}}{S_{land}} + \frac{\sqrt{EB^2 + (q - x)^2}}{S_{water}}$$

- Looks imposing, doesn't it? But it's really not *that* bad. It's just two distances, divided by their respective speeds, added together.
- So if we want the time to be a minimum, all we have to do (he says blithely!) is to differentiate it, set that to zero, and see what we come up with.
- Let's pause for, say, 30 seconds, while you think about this. Convince yourself that, yeah, that those are the correct formulas for distance and time.



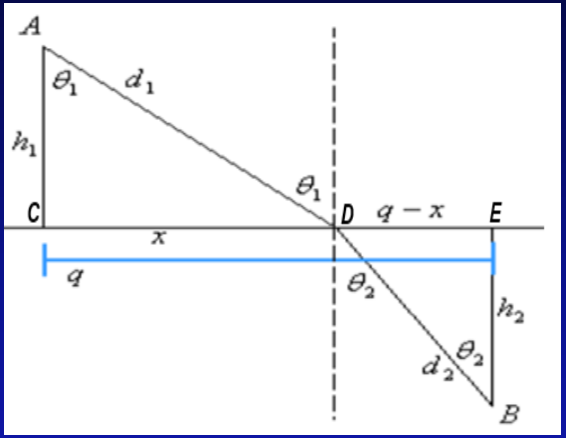
Setting Things Up - 3

The total time it takes to get to the distressed swimmer is

$$\frac{\sqrt{x^2 + AC^2}}{S_{land}} + \frac{\sqrt{EB^2 + (q - x)^2}}{S_{water}}$$

So that you shouldn't be *too* overwhelmed by all those symbols, let's plug in a few values. AC = 50; q = 300; EB = 30; S_{land} = 5; S_{water} = 1

$$\frac{\sqrt{x^2 + 2500}}{5} + \frac{\sqrt{900 + (300 - x)^2}}{1}$$



The Plot Thickens

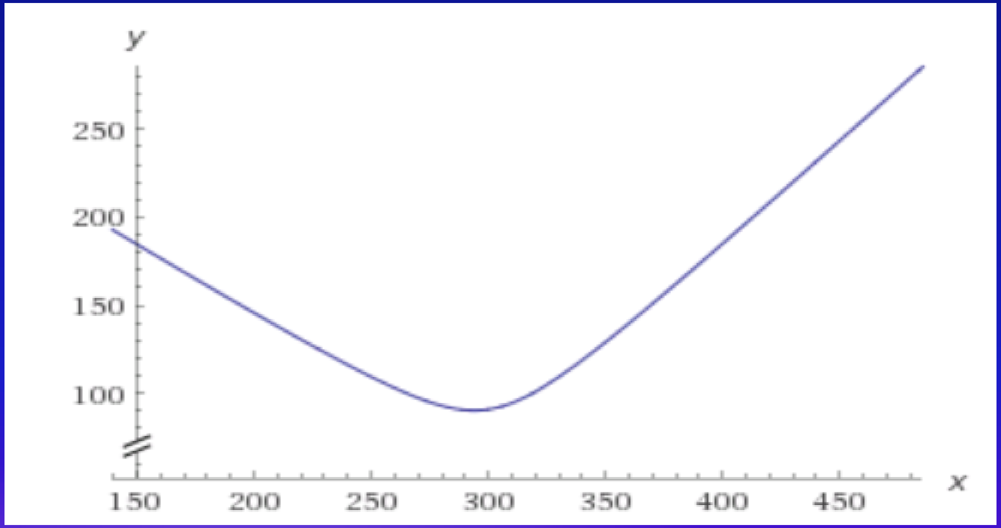
We want to plot $\frac{\sqrt{x^2 + 2500}}{5} + \frac{\sqrt{900 + (300 - x)^2}}{1}$

That's easy. Go to www.WolframAlpha.com and type in

plot (sqrt(x^2 + 2500) / 5) + sqrt(900 + (300 - x)^2)

And we get the minimum time just short of 300, as we've just calculated.

And notice that the point of least time, the slope of the curve is zero!



So Let's Do This

Remember that cheat sheet of derivatives from several slides back?

Differentiating $\frac{\sqrt{x^2+2500}}{5} + \frac{\sqrt{900+(300-x)^2}}{1}$ gives us…

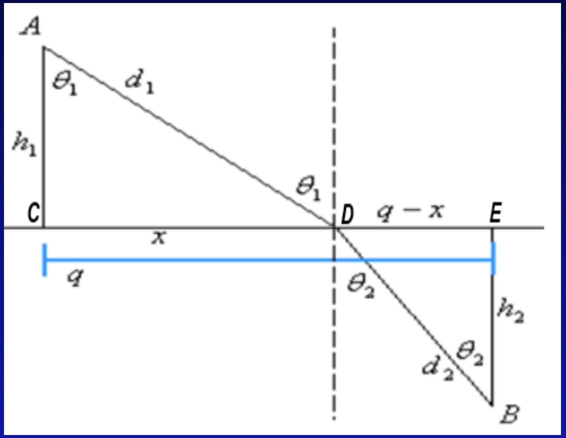
$$\frac{x}{5 * \sqrt{x^2+2500}} + \frac{-(300-x)}{\sqrt{900+(300-x)^2}}$$

Setting this to zero gives us

$$\frac{x}{5 * \sqrt{x^2+2500}} + \frac{-(300-x)}{\sqrt{900+(300-x)^2}} = 0$$

Solving this for *x* is a straightforward but messy bit of algebra.

Fortunately there's an easy way to solve it.



Function	Derivative
f(x) ± g(x)	f'(x) ± g'(x)
c * f(x)	c * f'(x)
$\sqrt{x^2+a^2}$	$\frac{x}{\sqrt{x^2+a^2}}$
$\sqrt{a^2+(q-x)^2}$	$\frac{-(q-x)}{\sqrt{a^2+(q-x)^2}}$

WolframAlpha to the Rescue Again

We want to solve $$\frac{x}{5 * \sqrt{x^2 + 2500}} + \frac{-(300 - x)}{\sqrt{900 + (300 - x)^2}} = 0$$

Go to www.wolframalpha.com and type in

solve (x / (5 * sqrt(x^2 + 2500))) - ((300 - x) / sqrt(900 + (300 - x)^2)) = 0

We get

Input interpretation:

solve $$\frac{x}{5\sqrt{x^2 + 2500}} - \frac{300 - x}{\sqrt{900 + (300 - x)^2}} = 0$$

Result:

$$x \approx 293.967$$

Well, that was easy!

One More Homework Problem

Or rather a set of them.

Play around with the constants in the scenario and use WolframAlpha to work out the numerical results. For example…

- What if the lifeguard's running speed were 3 m/sec? 2 m/sec? 1 m/sec? (think about this last one)
- What if the lifeguard's swimming speed were 2 m/sec? 5 m/sec? (think about this last one)
- Move the initial positions of the lifeguard and swimmer closer or farther from the shore. Or farther/closer in an east/west direction.

And can you figure out the angles (θ_1 and θ_2) for the original example, plus all the changes above?

Bottom Line (for now)

So the bottom line is that the lifeguard should run diagonally to the water line, 293.967 meters east of his original position. He should then swim directly to the ailing swimmer. That will get him there the quickest.

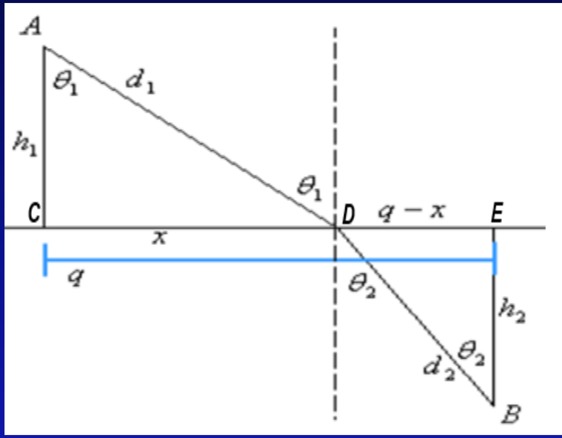
So I know what you're thinking. That was a lot of work, involving very (very!) messy expressions, with symbolic distances and time, involving squares, square roots, sub-expressions, addition, division, differentiating, setting things to zero, and solving equations.

But look on the bright side. At least we didn't also have to sacrifice a goat at dawn to get this to work! (I hate getting up early, don't you?)

Anyway, I've tried to show how you can take this semi-complex problem and break it into (hopefully comprehensible) pieces.

The differentiation process was perhaps the most straightforward of all, just looking up functions (e.g. $\sqrt{a^2 + x^2}$) and replacing them with their derivatives.

But we did manage to get a mathematically proven answer, so maybe it was worth the work.



I *Could* End the Talk Here

But I'd be leaving out the most important part!

The Most Important Part

OK, so we figured out where to enter the water. But there's a major extra piece of physics we can extract from all our setup work.

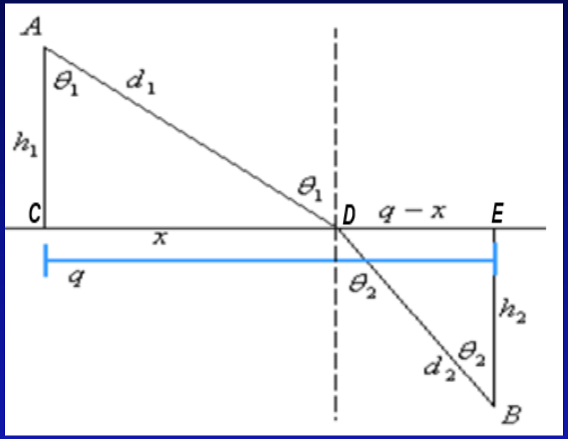
Let's go back to our original expression for the time it took to reach the swimmer:

$$\frac{\sqrt{x^2 + AC^2}}{S_{land}} + \frac{\sqrt{EB^2 + (q - x)^2}}{S_{water}}$$

Differentiate this and equate it to zero to get:

$$\frac{x}{S_{land} * \sqrt{x^2 + AC^2}} + \frac{-(q - x)}{S_{water} * \sqrt{EB^2 + (q - x)^2}} = 0$$

But $\frac{x}{\sqrt{x^2 + AC^2}}$ is just $\sin \theta_1$ and $\frac{(q - x)}{\sqrt{EB^2 + (q - x)^2}}$ is just $\sin \theta_2$



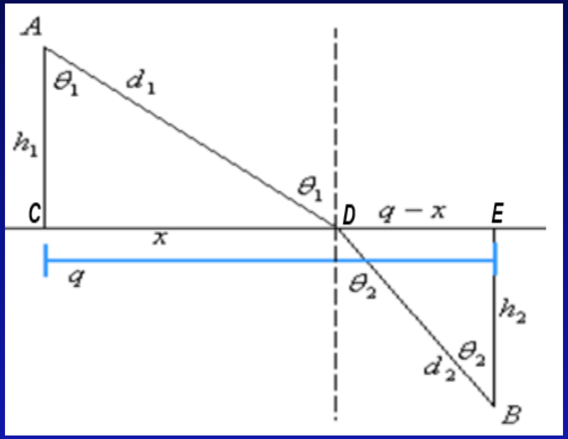
Snell's Law
The Law of Refraction

So $$\frac{x}{S_{land} * \sqrt{x^2 + AC^2}} + \frac{-(q - x)}{S_{water} * \sqrt{EB^2 + (q - x)^2}} = 0$$

becomes $$\frac{\sin \theta_1}{S_{land}} + \frac{-\sin \theta_2}{S_{water}} = 0$$

or $$\frac{\sin \theta_1}{S_{land}} = \frac{\sin \theta_2}{S_{water}}$$

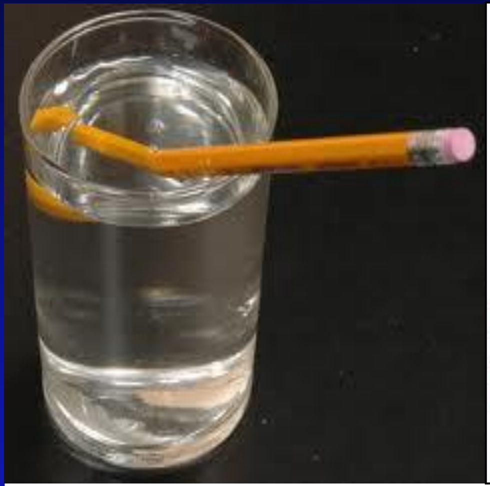
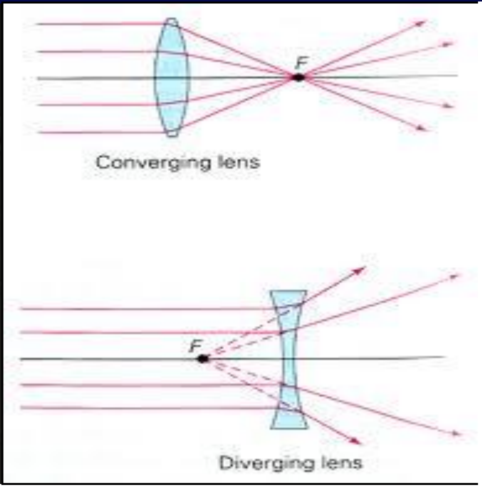
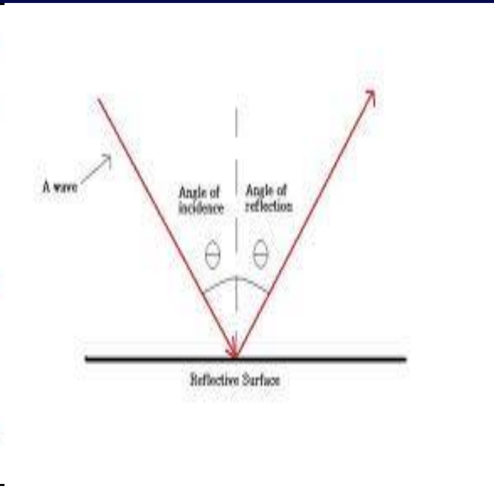
Alternatively we can write $$\frac{\sin \theta_1}{\sin \theta_2} = \frac{S_{land}}{S_{water}}$$



In words, if the transit time is minimized when travelling through two media (e.g. the passage of light), the angle of refraction is determined by the propagation speeds involved, according to this formula.

This result is called Snell's Law.

What It Explains (among others)

		
Why primitive spear fishermen had to aim where the fish *wasn't*!	How your eyeglasses work.	Put the swimmer back on land, so that the speed in the "two" media are the same, and we get Angle of Incidence = Angle of Reflection

Wrapping Up

We've only scratched the surface. And that's just the surface of one branch of calculus.

But we've accomplished a *lot* in the last hour or two.

Hopefully you now *understand* a bit of what Differential Calculus is and how it can be used.

Thank you

References

- http://en.wikipedia.org/wiki/Limit_(mathematics)
- http://en.wikipedia.org/wiki/Epsilon,_delta
- http://en.wikipedia.org/wiki/Snell%27s_law
- http://en.wikipedia.org/wiki/Fermat%27s_principle
- http://www.khanacademy.org